Resurrection of the Limit Analysis in Geotechnical Design

Christian Olsen

ABSTRACT

The most recent publication of the offshore geotechnical design codes from both DNV and ISO have included limit analysis as an alternative design method to the bearing capacity formulation, limit equilibrium and Finite Element Analysis FEA. In the past, limit analysis was not mentioned in the offshore geotechnical design codes. In this paper, a brief introduction to limit equilibrium, limit analysis and FEA is presented. In addition, some newly developed software based on Finite Element Limit Analysis are discussed.

1. INTRODUCTION

To obtain the correct solution for the collapse of a foundation, three conditions must be satisfied, see Ref. 2:

1. Static condition – equilibrium between all stresses
2. Kinematic condition – solution must be compatible
3. Stress –strain relationship – connects 1. and 2.

It is for most cases impossible to satisfy all three conditions and some methods have been based on satisfying two. The following paper sets out to describe some of these methods. The first method is the limit equilibrium followed by the limit analysis.

This paper is intended as a brief introduction to understand the difference between limit equilibrium, limit analysis and also to get some insight into FEA and FELA. However, the paper is too brief to get the full overview and details are left out.

2. LIMIT EQUILIBRIUM

Limit Equilibrium first introduced by Coulomb in 1773 is roughly described as defining a slip line and then determine static equilibrium along the slip line. An example is given below for the modified Bishops method, see Figure 2-1.

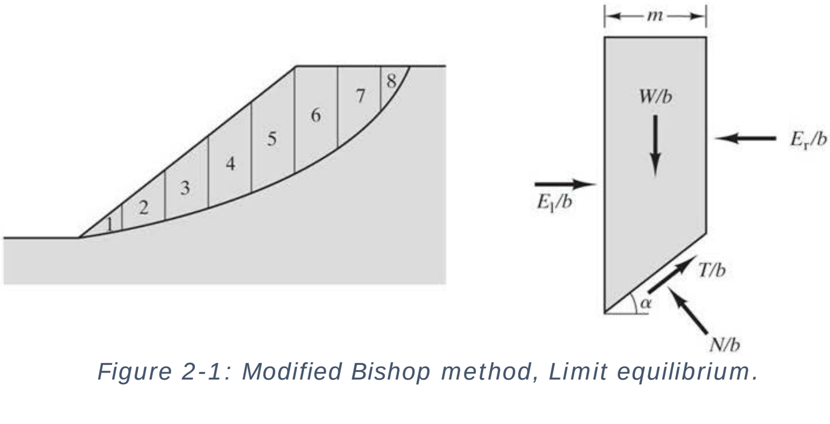

Figure 2-1: Modified Bishop method, Limit equilibrium.

There are some problems with the method when it comes to satisfying the three conditions required for the correct collapse load or failure mechanism defined in Section 1. These are:

- Theory does not consider strain and displacement compatibility.
- not all equations of statics are satisfied, see Table 2-1
- No simple means of checking accuracy

Method	Moment equilibrium	Horizontal force equilibrium	Interslice normal (E)	Interslice shear (X)
Ordinary or Fellenius	Yes	No	No	No
Bishop's simplified	Yes	No	Yes	No
Janbu's simplified	No	Yes	Yes	No
Spencer	Yes	Yes	Yes	Yes
Morgenstern–Price	Yes	Yes	Yes	Yes
Corps of Engineers – 1	No	Yes	Yes	Yes
Corps of Engineers – 2	No	Yes	Yes	Yes
Lowe–Karafiath	No	Yes	Yes	Yes

Table 2-1: Limitations of limit equilibrium Ref. 1

At the International Symposium on Limit Analysis at Tsinghua University, Kristian Krabbenhøft presented a comparison between eight limit equilibrium methods to limit analysis. The limit equilibrium methods are given below in Table 2-2.

Method	Slip surface
1. Janbu Simplified	Non-circular
2. Spencer	Circular
3. Janbu Corrected	Non-circular
4. Spencer	Non-circular
5. Janbu Corrected	Circular
6. Bishop	Circular
7. GLE	Circular
8. Bishop Simplified	Circular

Table 2-2: Limit equilibrium methods.

The slope problem used as a benchmark is for a 20 m high slope with the drained properties as given in Figure 2-2.

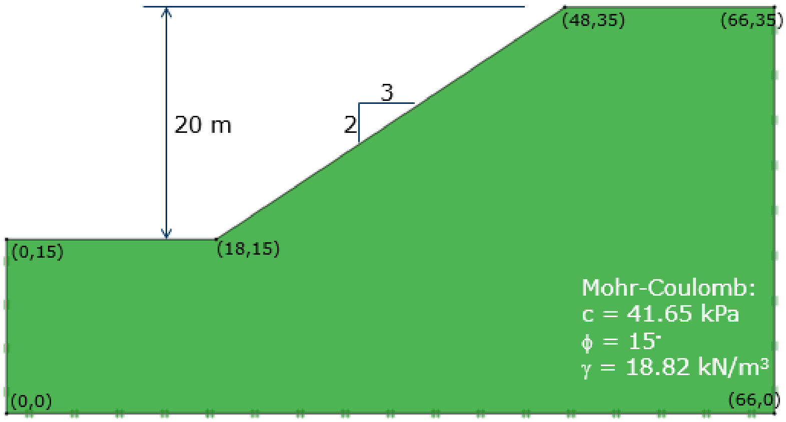

Figure 2-2: Benchmark case for comparing limit equilibrium and limit analysis.

The results of the eight limit equilibrium methods are given as a factor of safety. The factor of safety is then compared to the upper and lower bound of the limit analysis. The comparison is given in Figure 2-3.

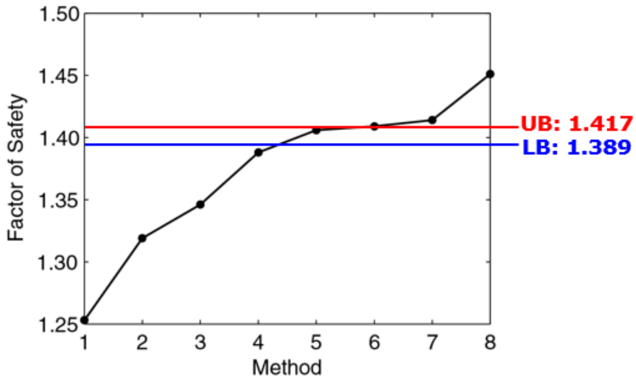

Figure 2-3: Comparison between limit equilibrium and limit analysis for slope stability.

The comparison indicates that some of the limit equilibrium methods gives estimates on the unsafe side while others are conservative. As indicated in the beginning, it is not possible with the limit equilibrium method to determine whether the design is safe or unsafe.

3. LIMIT ANALYSIS

Drucker et al. 1951 originally develop the limit analysis. Lower and upper bounds define the limit analysis method as follows:

- Lower bound: Equilibrium (static admissible), and yield criterion satisfied
- Upper bound: Kinematically admissible, plastically admissible (flow rule)

The static admissible solution is always on the safe side; hence, it is called a lower bound. The better the guess of a possible stress field, the closer the result will be to the correct solution. The practical application of the static admissible solutions method is:

1. A possible stress field is identified in the ground
2. All stresses are accounted for, so yield is not exceeded at any one location
3. The load is found by equilibrium

4. Consider to define a parameter which can be varied to identify the extreme maximum value

The possibly simplest and most referenced case of a lower bound solution is given in Figure 3-1. The static admissible solution “proves” that 4*su is a lower bound estimate of the bearing capacity for clay with a constant undrained shear strength.

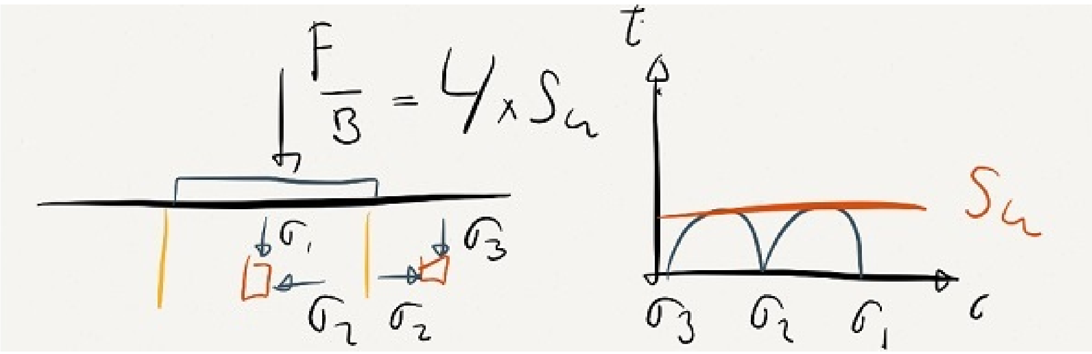

Figure 3-1: Static admissible solution

The kinematically admissible solution is always on the unsafe side; hence, it is called an upper bound. The better the guess is of a possible kinematically solution, the closer the result will be to the correct solution. The practical application of the kinematically admissible solutions method is:

1. A possible failure mechanism is identified
2. All displacements are accounted for in the failure mechanism
3. Work equation is written up based on a finite displacement.
4. Consider to define a parameter which can be varied to identify the extreme-minimum value

In Figure 3-2, an example of a kinematically admissible solution is given. Here the parameter to be varied is the angle θ. When minimizing the work equation for θ, then it is possible for each α to determine the failure mechanism with least energy.

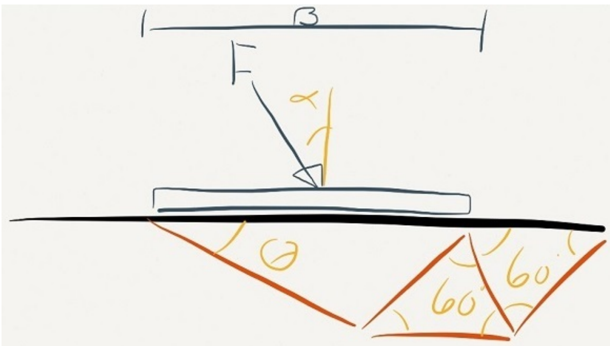

Figure 3-2: Kinematically admissible solution

The result is an upper bound solution as a function of α and when plotted in the horizontal and vertical load diagram, the well know stability envelope appears, see Figure 3-3. The above problem is described in more detail on the webpage http://colsen.dk/geotechnics.html regarding the kinematically admissible solution.

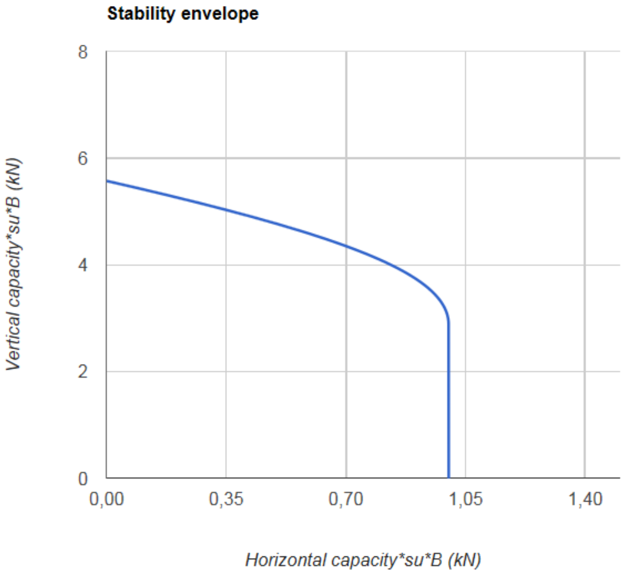

Figure 3-3: Stability envelope based on a kinematically admissible solution

4. DESIGN CODES

Karl Terzaghi proposed the bearing capacity equation, which is probably the most used and least fully understood equation in geotechnical engineering. The equation is based on superposition of the effects from the soil weight, the soil cohesion and a surface load.

$$R = \frac{1}{2}\gamma b^2 N_\gamma + qbN_q + cbN_c$$

Then Brinch Hansen used limit analysis to determine the factors in the Karl Terzaghi bearing capacity equation, examples of the static and kinematically admissible failure mechanisms are given below in Figure 4-1.

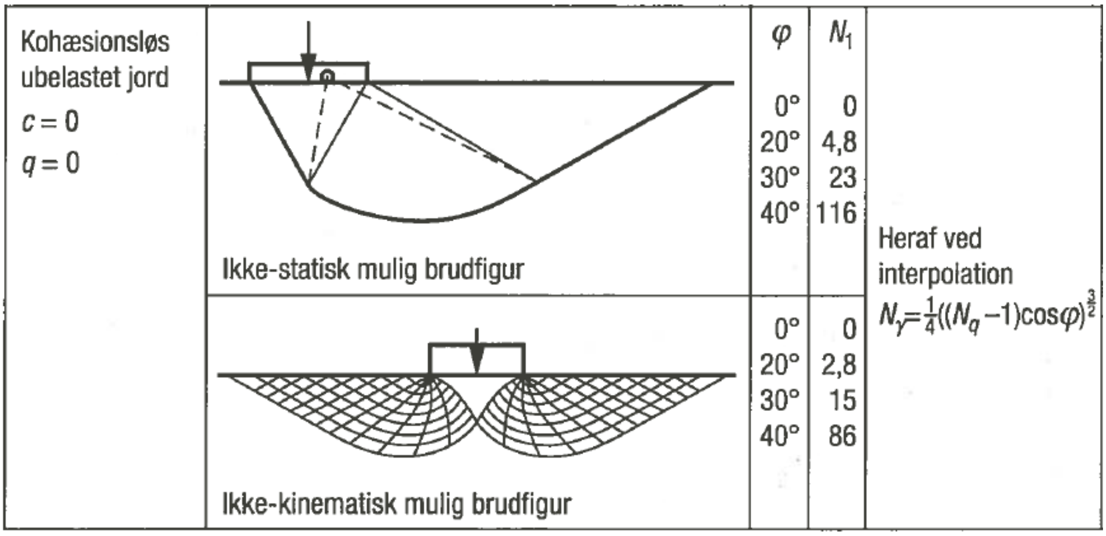

Figure 4-1: Brinch Hansen static and kinematically admissible failure mechanisms.

The bearing capacity equation has since been adopted in probably all design codes for geotechnical engineering such as DNV classification notes 30.4, the Chineese code GB 50007-2002, ISO 19901-4 2003 and many more.

Despite the bearing capacity equation is defined by limit analysis, then limit analysis was not mentioned in the DNV CN 30.4 and the previous version of ISO 19901-4. However, in 2016 ISO issued the new revision of the ISO 19901-4 and in 2017 DNVGL issued the new foundation code DNVGL-RP-C212. Both these codes now proposes that alternatives to the bearing capacity equation are limit analysis, limit equilibrium and FEA. There can be many reasons why the limit analysis was left out in the previous codes. Whatever the reason was, then this is why this paper is called the resurrection of limit analysis.

5. FINITE ELEMENT ANALYSIS

The finite element analysis has been getting increasingly more used over the last 30 years. It is today probably viewed as the most accurate method to solve complex problems. Despite of this, I do not necessarily agree with the popular view on this.

For example, the PLAXIS 3D software is based on a strain element type (kinematically admissible) as written in section 2.1 in the scientific manual. It is written that the displacements are solved with a kinematically admissible variation of displacements. The design process of a FEA is given in Figure 5-1. In the Figure, it is possible to see that increased mesh density will lower the failure load toward the exact solution.

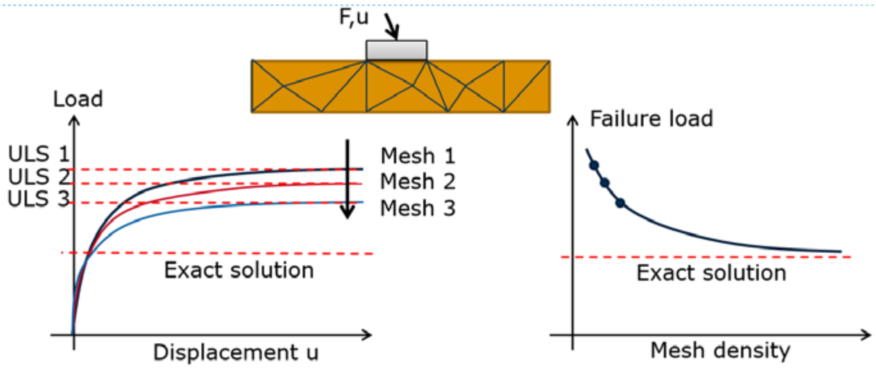

Figure 5-1: Design process for FEA, courtesy of OptumCE

Since the FEA normally is based on kinematically admissible displacements, then the result is an upper bound approach. Therefore, the results from PLAXIS should always be on the unsafe side for collapse design – unless the model did fail to complete the calculation.

Another challenge with FEA is that to get the failure load, the full load-displacement curve is needed to determine the ULS load. Further, there is no easy way to evaluate the quality of the result. A convergence analysis is probably the only solution to assess the quality of the result. This means that several load-displacement curves needs to be established which makes FEA time consuming.

6. FINITE ELEMENT LIMIT ANALYSIS

Another reason why this paper is called resurrection of limit analysis is that in recent years, several engineers have programmed finite element limit analysis software. A great example is the OxLIM software, developed by Chris Martin. A phd has been written based on OxLIM, see Ref. 3.

When FELA contains both the upper and lower bound, it is possible to get an understanding of the quality of the failure load. This is illustrated in Figure 6-1

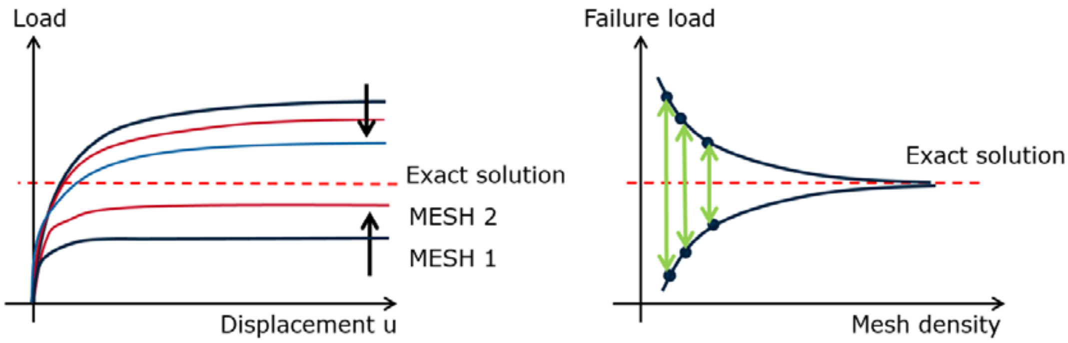

Figure 6-1: Design process for FELA, courtesy of OptumCE

Current commercial available FELA is OptumG2 and OptumG3, see the https://optumce.com/ for further details. These have been developed over the past years and are very user-friendly. I have completed a review of OptumG3 and OptumG2. FELA does not calculate the load-displacement curve, but only the collapse load. This makes the method much faster than FEA. Finally, since the designer can get both lower and upper bound, this is possibly the only method to assess the quality of the results.

7. CONCLUSION

Limit analysis and limit equilibrium is two different geotechnical engineering methods, which at first may seem similar. Regarding the accuracy of the limit equilibrium method, then it is not always possible to understand whether the result is an under- or overestimate of the capacity. The limit analysis will bound the collapse load and therefore the method allows for an assessment of the quality of the result.

Limit analysis was left out of the typical offshore geotechnical design codes such as DNV and ISO. Both these have recently been updated in ~2016 to also include limit analysis in addition to the limit equilibrium and FEA.

If used correct, FEA is an upper bound; however, it will likely convert close to or at the correct solution. To assess the quality of the result, a sensitivity study of the mesh density should be completed. This will require several runs and the generation of many “unnecessary” load-displacement curves. Unnecessary since the definition of the collapse load does not depend of the elastic stiffness of the soil.

FELA is a finite element approach to limit analysis. For ultimate capacity design, then the FELA makes great sense. First of all, the geotechnical engineer will be able to bound the problem. Hence, FELA is the only method, which will provide an estimate that can be “guaranteed” as accurate. Also, FELA is much faster than FEA since the full load-displacement curve is not generated.

8. REFERENCES

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